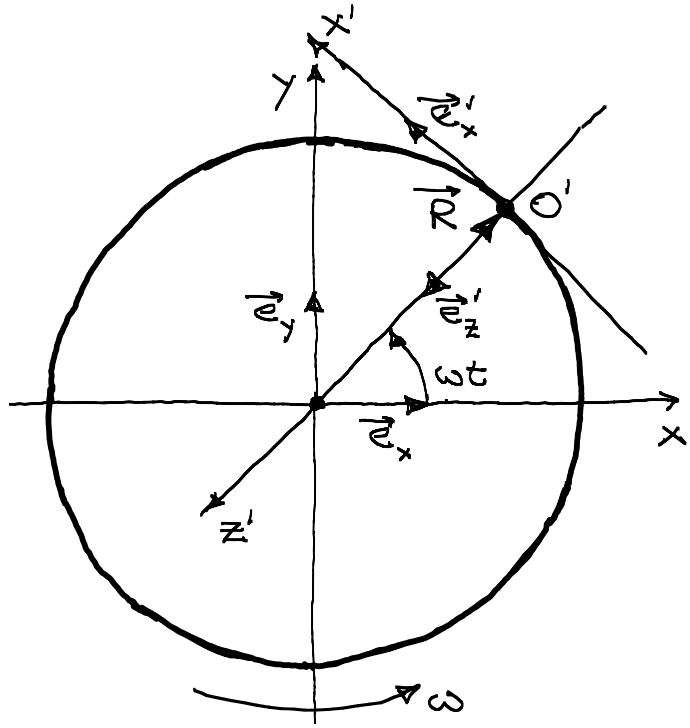
1



$\vec{e}_z$ out of page
$\vec{e}_y'$ into page

We will need

$$\vec{e}_x' = -\vec{e}_x \sin\omega t + \vec{e}_y \cos\omega t$$
$$\vec{e}_y' = -\vec{e}_z$$
$$\vec{e}_z' = -\vec{e}_x \cos\omega t - \vec{e}_y \sin\omega t$$

and
$$x' = -x\sin\omega t + y\cos\omega t$$
$$y' = -z$$
$$z' = R - x\cos\omega t - y\sin\omega t$$

$$\vec{e}_j' = A_{jk}\vec{e}_k \qquad A = \begin{pmatrix} -s & c & 0 \\ 0 & 0 & -1 \\ -c & -s & 0 \end{pmatrix}$$

$$x_k\vec{e}_k = \vec{R} + x_j'\vec{e}_j' = (x_j' - R\delta_{jz})\vec{e}_{j'} = (x_j' - R\delta_{jz})A_{jk}\vec{e}_k$$
$$\Rightarrow x_k = (x_j' - R\delta_{jz})A_{jk}$$
or $(x \;\; y \;\; z) = (x' \;\; y' \;\; z'-R)\,A$

or $(x' \;\; y' \;\; z'-R) = (x \;\; y \;\; z)\,A^T$

$$\begin{pmatrix} x' \\ y' \\ z'-R \end{pmatrix} = A\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

(a)
$$\boxed{\begin{aligned} \vec{\omega} &= \omega\vec{e}_z = -\omega\vec{e}_y' \\ \vec{R} &= -R\vec{e}_z' = R(\cos\omega t\,\vec{e}_x + \sin\omega t\,\vec{e}_y) \\ \dot{\vec{R}} &= \omega R(-\sin\omega t\,\vec{e}_x + \cos\omega t\,\vec{e}_y) = \vec{\omega}\times\vec{R} = \omega R\vec{e}_x' \\ \ddot{\vec{R}} &= \omega^2 R(-\cos\omega t\,\vec{e}_x - \sin\omega t\,\vec{e}_y) = \vec{\omega}\times(\vec{\omega}\times\vec{R}) = \omega^2 R\vec{e}_z' \end{aligned}}$$

(b) Method 1: Use the equations for a rotating frame

$$\vec{a}_r = \underbrace{-\vec{\omega}\times(\vec{\omega}\times\vec{R})}_{-\omega^2 R\vec{e}_z'} - \vec{\omega}\times(\vec{\omega}\times\vec{r}) - \underbrace{2\vec{\omega}\times\vec{v}_r}$$

main centrifugal acceleration

$$2\omega\vec{e}_{y'}\times\vec{v}_r = -2\omega\dot{x}'\vec{e}_{z'} + 2\omega\dot{z}'\vec{e}_{x'}$$

Coriolis force

$$-\omega^2\vec{e}_y'\times(\vec{e}_y'\times\vec{r}) = -\omega^2\vec{e}_y'\times(-x'\vec{e}_z' + z'\vec{e}_{x'})$$

$$= \omega^2 x'\vec{e}_x' + \omega^2 z'\vec{e}_z'$$

additional centrifugal acceleration when not at O'

$$\boxed{\begin{aligned} \ddot{x}' &= \omega^2 x' + 2\omega\dot{z}' \\ \ddot{z}' &= -\omega^2 R + \omega^2 z' - 2\omega\dot{x}' \\ \ddot{y}' &= 0 \end{aligned}}$$

Method 2. Write the primed coördinates in terms of the inertial coördinates.

$$x' = -x\sin\omega t + y\cos\omega t$$

$$y' = -z$$

$$z' = R - x\cos\omega t - y\sin\omega t$$

The inertial coördinates satisfy

$$\ddot{x} = \ddot{y} = \ddot{z} = 0,$$

so

$$\dot{x}' = -\dot{x}s + \dot{y}c - \omega\overbrace{(xc+ys)}^{R-z'}$$

$$\ddot{x}' = -2\omega\underbrace{(\dot{x}c+\dot{y}s)}_{-\omega x'-\dot{z}'} + \omega^2\underbrace{(xs-yc)}_{-x'}$$

$$\dot{z}' = -\dot{x}c - \dot{y}s + \omega\overbrace{(xs-yc)}^{-x'}$$

$$\ddot{z}' = 2\omega\underbrace{(\dot{x}s-\dot{y}c)}_{-\omega(R-z')-\dot{x}'} + \omega^2\underbrace{(xc+ys)}_{R-z'}$$

$$\Longrightarrow \boxed{\begin{aligned} \ddot{x}' &= \omega^2 x' + 2\omega\dot{z}' \\ \ddot{z}' &= -\omega^2(R-z') - 2\omega\dot{x}' \\ \ddot{y}' &= 0 \end{aligned}}$$

(c) Initial conditions: $x'(0)=0$, $\dot{x}'(0)=0$, $z'(0)=h$, $\dot{z}'(0)=0$

Method 1. One neglects the tiny centrifugal corrections $\omega^2 x'$ and $\omega^2 z'$. In addition, one neglects the Coriolis term in the z' equation. So

$$\ddot{z}' = -\omega^2 R \quad \Rightarrow \quad \dot{z}' = -\omega^2 R t \quad \text{and} \quad z' = h - \frac{1}{2}\omega^2 R t^2$$

fall under effective gravity $g = \omega^2 R$

$$\ddot{x}' = 2\omega \dot{z}' = -2\omega^3 R t \Rightarrow x' = -\frac{1}{3}R(\omega t)^3$$

$$\boxed{\begin{aligned} z'(t) &= h - \frac{1}{2}R(\omega t)^2 \\ x' &= -\frac{1}{3}R(\omega t)^3 \end{aligned}}$$

Method 2. One can easily find the exact solution in the inertial coördinates.

Initial conditions: $x(0) = R-h$, $\dot{x}(0)=0$, $y(0)=0$, $\dot{y}(0) = \omega(R-h)$

$$\Rightarrow \quad x(t) = R-h \quad \text{and} \quad y(t) = \omega t(R-h)$$

Exact: $$x'(t) = (R-h)\underbrace{\left(-\sin\omega t + \omega t\cos\omega t\right)}_{\omega t \ll 1:\ -\omega t + \frac{1}{6}(\omega t)^3 + \omega t - \frac{1}{2}(\omega t)^3 = -\frac{1}{3}(\omega t)^3}$$

Exact: $$z'(t) = R - (R-h)\underbrace{\left(\cos\omega t + \omega t \sin\omega t\right)}_{\omega t \ll 1:\ 1 - \frac{1}{2}(\omega t)^2 + (\omega t)^2 = 1 + \frac{1}{2}(\omega t)^2}$$

$$\omega t \ll 1: \quad z' = h - \frac{1}{2}(R-h)(\omega t)^2 = h - \frac{1}{2}R(\omega t)^2$$

neglect

$$x' = -\frac{1}{3}(R-h)(\omega t)^3 = -\frac{1}{3}R(\omega t)^3$$

$$\boxed{\begin{aligned} z'(t) &= h - \frac{1}{2}R(\omega t)^2 \\ x' &= -\frac{1}{3}R(\omega t)^3 \end{aligned}}$$

The particle hits the ground at $t = \sqrt{2h/g}$, $g = \omega^2 R$, when

$$x' = -\frac{1}{3} R\omega^3 t^3 = -\frac{1}{3} g \sqrt{\frac{g}{R}} \frac{2h}{g} \sqrt{\frac{2h}{g}} = -\frac{2\sqrt{2}}{3}\sqrt{\frac{h}{R}}\, h,$$

so

$$\frac{x'}{h} = -\frac{2\sqrt{2}}{3}\underbrace{\sqrt{\frac{h}{R}}}_{\simeq \frac{\sqrt{2}}{100}} \simeq -\frac{4}{3} \times 0.01 \Longrightarrow$$

$|x'|$ would be about 2 cm, so you could see this in a careful measurement.

$h \simeq 2\,m$
$R = 10^4\,m$