General Relativity, Einstein & All That (GREAT)

First, load the package

In[32]:=
```
<< GREAT.m
```

```
GREAT functions are: IMetric, Christoffel,
  Riemann, Ricci, SCurvature, EinsteinTensor, SqRicci, SqRiemann.
```

```
Enter 'helpGREAT' for this list of functions
```

Brief on-line help is available for all functions:

```
? IMetric
```

IMetric[g], with g an n.n-matrix (with lower indices), returns the inverse (with upper indices).

A sample calculation

First define the coordinate *n*-vector:

In[42]:=
```
X = { t, r, θ, ϕ}
```

Out[42]= {t, r, θ, ϕ}

and then specify the metric as a square $n \times n$ matrix:

In[96]:=
$$\left(met = \left\{\{-1, 0, 0, 0\}, \left\{0, \frac{a[t]^2}{1-(k[t])\,r^2}, 0, 0\right\}, \left\{0, 0, a[t]^2\,r^2, 0\right\}, \left\{0, 0, 0, a[t]^2\,r^2\,(Sin[\theta])^2\right\}\right\}\right) \text{ // MatrixForm}$$

Out[96]//MatrixForm=
$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a[t]^2}{1-r^2\,k[t]} & 0 & 0 \\ 0 & 0 & r^2\,a[t]^2 & 0 \\ 0 & 0 & 0 & r^2\,a[t]^2\,Sin[\theta]^2 \end{pmatrix}$$

"**IMetric**" is the only 1-argument function:

In[98]:=
```
IMetric[met] // MatrixForm
```

Out[98]//MatrixForm=
$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1-r^2\,k[t]}{a[t]^2} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2\,a[t]^2} & 0 \\ 0 & 0 & 0 & \frac{Csc[\theta]^2}{r^2\,a[t]^2} \end{pmatrix}$$

All other "functions" take two arguments, the metric matrix and then the coordinate vector:

In[99]:= **Christoffel[met, x]**

Out[99]=
$$\left\{\left\{\{0, 0, 0, 0\}, \left\{0, \frac{a[t]\left(\left(2-2r^2k[t]\right)a'[t]+r^2a[t]k'[t]\right)}{2\left(-1+r^2k[t]\right)^2}, 0, 0\right\},\right.\right.$$
$$\left.\left\{0, 0, r^2a[t]a'[t], 0\right\}, \left\{0, 0, 0, r^2a[t]\text{Sin}[\theta]^2a'[t]\right\}\right\},$$
$$\left\{\left\{0, \frac{2a'[t]-2r^2k[t]a'[t]+r^2a[t]k'[t]}{2a[t]-2r^2a[t]k[t]}, 0, 0\right\},\right.$$
$$\left\{\frac{2a'[t]-2r^2k[t]a'[t]+r^2a[t]k'[t]}{2a[t]-2r^2a[t]k[t]}, \frac{r\,k[t]}{1-r^2k[t]}, 0, 0\right\},$$
$$\left.\left\{0, 0, r\left(-1+r^2k[t]\right), 0\right\}, \left\{0, 0, 0, r\left(-1+r^2k[t]\right)\text{Sin}[\theta]^2\right\}\right\},$$
$$\left\{\left\{0, 0, \frac{a'[t]}{a[t]}, 0\right\}, \left\{0, 0, \frac{1}{r}, 0\right\}, \left\{\frac{a'[t]}{a[t]}, \frac{1}{r}, 0, 0\right\}, \{0, 0, 0, -\text{Cos}[\theta]\text{Sin}[\theta]\}\right\},$$
$$\left.\left\{\left\{0, 0, 0, \frac{a'[t]}{a[t]}\right\}, \left\{0, 0, 0, \frac{1}{r}\right\}, \{0, 0, 0, \text{Cot}[\theta]\}, \left\{\frac{a'[t]}{a[t]}, \frac{1}{r}, \text{Cot}[\theta], 0\right\}\right\}\right\}$$

In[100]:= **Riemann[met, x]**

Out[100]=
$$\left\{\left\{\{\{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\},\right.\right.$$
$$\left\{\left\{0, -\left(a[t]\left(-4r^2\left(-1+r^2k[t]\right)a'[t]k'[t]+4\left(-1+r^2k[t]\right)^2a''[t]+\right.\right.\right.\right.$$
$$\left.\left.\left.a[t]\left(3r^4k'[t]^2-2r^2\left(-1+r^2k[t]\right)k''[t]\right)\right)\right)\Big/\left(4\left(-1+r^2k[t]\right)^3\right), 0, 0\right\},$$
$$\left\{\left(a[t]\left(-4r^2\left(-1+r^2k[t]\right)a'[t]k'[t]+4\left(-1+r^2k[t]\right)^2a''[t]+\right.\right.\right.$$
$$\left.\left.\left.a[t]\left(3r^4k'[t]^2-2r^2\left(-1+r^2k[t]\right)k''[t]\right)\right)\right)\Big/\left(4\left(-1+r^2k[t]\right)^3\right), 0, 0, 0\right\},$$
$$\left.\{0, 0, 0, 0\}, \{0, 0, 0, 0\}\right\}, \left\{\left\{0, 0, r^2a[t]a''[t], 0\right\}, \left\{0, 0, \frac{r^3a[t]^2k'[t]}{-2+2r^2k[t]}, 0\right\},\right.$$
$$\left.\left\{-r^2a[t]a''[t], \frac{r^3a[t]^2k'[t]}{2-2r^2k[t]}, 0, 0\right\}, \{0, 0, 0, 0\}\right\},$$
$$\left\{\left\{0, 0, 0, r^2a[t]\text{Sin}[\theta]^2a''[t]\right\}, \left\{0, 0, 0, \frac{r^3a[t]^2\text{Sin}[\theta]^2k'[t]}{-2+2r^2k[t]}\right\},\right.$$
$$\left.\left.\{0, 0, 0, 0\}, \left\{-r^2a[t]\text{Sin}[\theta]^2a''[t], \frac{r^3a[t]^2\text{Sin}[\theta]^2k'[t]}{2-2r^2k[t]}, 0, 0\right\}\right\}\right\},$$
$$\left\{\left\{\left\{0, \left(-4r^2\left(-1+r^2k[t]\right)a'[t]k'[t]+4\left(-1+r^2k[t]\right)^2a''[t]+\right.\right.\right.\right.$$
$$\left.\left.a[t]\left(3r^4k'[t]^2-2r^2\left(-1+r^2k[t]\right)k''[t]\right)\right)\Big/\left(4a[t]\left(-1+r^2k[t]\right)^2\right), 0, 0\right\},$$
$$\left\{\left(4r^2\left(-1+r^2k[t]\right)a'[t]k'[t]-4\left(-1+r^2k[t]\right)^2a''[t]+\right.\right.$$
$$\left.\left.a[t]\left(-3r^4k'[t]^2+2r^2\left(-1+r^2k[t]\right)k''[t]\right)\right)\Big/\left(4a[t]\left(-1+r^2k[t]\right)^2\right), 0, 0, 0\right\},$$
$$\left.\{0, 0, 0, 0\}, \{0, 0, 0, 0\}\right\}, \{\{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\},$$
$$\left\{\left\{0, 0, \frac{1}{2}r^3k'[t], 0\right\}, \left\{0, 0, \frac{1}{-2+2r^2k[t]}\right.\right.$$
$$r^2\left(2r^2k[t]^2+2k[t]\left(-1+r^2a'[t]^2\right)-a'[t]\left(2a'[t]+r^2a[t]k'[t]\right)\right),$$
$$\left.0\right\}, \left\{-\frac{1}{2}r^3k'[t], \frac{1}{-2+2r^2k[t]}\right.$$
$$\left.r^2\left(-2r^2k[t]^2+k[t]\left(2-2r^2a'[t]^2\right)+a'[t]\left(2a'[t]+r^2a[t]k'[t]\right)\right), 0, 0\right\},$$
$$\left.\{0, 0, 0, 0\}\right\}, \left\{\left\{0, 0, 0, \frac{1}{2}r^3\text{Sin}[\theta]^2k'[t]\right\}, \left\{0, 0, 0, \frac{1}{-2+2r^2k[t]}\right.\right.$$
$$\left.r^2\text{Sin}[\theta]^2\left(2r^2k[t]^2+2k[t]\left(-1+r^2a'[t]^2\right)-a'[t]\left(2a'[t]+r^2a[t]k'[t]\right)\right)\right\},$$

$$\{0,0,0,0\},\left\{-\frac{1}{2}r^3\,\mathrm{Sin}[\theta]^2\,k'[t],\ \frac{1}{-2+2r^2\,k[t]}\right.$$

$$\left.r^2\,\mathrm{Sin}[\theta]^2\left(-2r^2\,k[t]^2+k[t]\left(2-2r^2\,a'[t]^2\right)+a'[t]\left(2a'[t]+r^2\,a[t]\,k'[t]\right)\right),0,0\right\}\Big\}\Big\},$$

$$\left\{\left\{\left\{0,0,\frac{a''[t]}{a[t]},0\right\},\left\{0,0,\frac{r\,k'[t]}{-2+2r^2\,k[t]},0\right\},\left\{-\frac{a''[t]}{a[t]},\frac{r\,k'[t]}{2-2r^2\,k[t]},0,0\right\},\right.\right.$$

$$\{0,0,0,0\}\Big\},\left\{\left\{0,0,\frac{r\,k'[t]}{-2+2r^2\,k[t]},0\right\},\right.$$

$$\left\{0,0,\frac{k[t]}{-1+r^2\,k[t]}-\frac{a'[t]\left(\left(2-2r^2\,k[t]\right)a'[t]+r^2\,a[t]\,k'[t]\right)}{2\left(-1+r^2\,k[t]\right)^2},0\right\},$$

$$\left\{\frac{r\,k'[t]}{2-2r^2\,k[t]},\frac{k[t]}{1-r^2\,k[t]}+\frac{a'[t]\left(\left(2-2r^2\,k[t]\right)a'[t]+r^2\,a[t]\,k'[t]\right)}{2\left(-1+r^2\,k[t]\right)^2},0,0\right\},$$

$$\{0,0,0,0\}\Big\},\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},$$

$$\left\{\{0,0,0,0\},\{0,0,0,0\},\left\{0,0,0,r^2\,\mathrm{Sin}[\theta]^2\left(k[t]+a'[t]^2\right)\right\},\right.$$

$$\left.\left\{0,0,-r^2\,\mathrm{Sin}[\theta]^2\left(k[t]+a'[t]^2\right),0\right\}\right\}\Big\},$$

$$\left\{\left\{\left\{0,0,0,\frac{a''[t]}{a[t]}\right\},\left\{0,0,0,\frac{r\,k'[t]}{-2+2r^2\,k[t]}\right\},\{0,0,0,0\},\right.\right.$$

$$\left\{-\frac{a''[t]}{a[t]},\frac{r\,k'[t]}{2-2r^2\,k[t]},0,0\right\}\Big\},\left\{\left\{0,0,0,\frac{r\,k'[t]}{-2+2r^2\,k[t]}\right\},\right.$$

$$\left\{0,0,0,\frac{k[t]}{-1+r^2\,k[t]}-\frac{a'[t]\left(\left(2-2r^2\,k[t]\right)a'[t]+r^2\,a[t]\,k'[t]\right)}{2\left(-1+r^2\,k[t]\right)^2}\right\},\{0,0,0,0\},$$

$$\left\{\frac{r\,k'[t]}{2-2r^2\,k[t]},\frac{k[t]}{1-r^2\,k[t]}+\frac{a'[t]\left(\left(2-2r^2\,k[t]\right)a'[t]+r^2\,a[t]\,k'[t]\right)}{2\left(-1+r^2\,k[t]\right)^2},0,0\right\}\Big\},$$

$$\left\{\{0,0,0,0\},\{0,0,0,0\},\left\{0,0,0,-r^2\left(k[t]+a'[t]^2\right)\right\},\left\{0,0,r^2\left(k[t]+a'[t]^2\right),0\right\}\right\},$$

$$\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\}\Big\}\Big\}$$

In[101]:= **Ricci[met, x]**

Out[101]=

$$\left\{\left\{\left(4r^2\left(-1+r^2\,k[t]\right)a'[t]\,k'[t]-12\left(-1+r^2\,k[t]\right)^2a''[t]+\right.\right.\right.$$

$$\left.a[t]\left(-3r^4\,k'[t]^2+2r^2\left(-1+r^2\,k[t]\right)k''[t]\right)\right)\Big/\left(4\,a[t]\left(-1+r^2\,k[t]\right)^2\right),\ \frac{r\,k'[t]}{1-r^2\,k[t]},$$

$$0,0\Big\},\left\{\frac{r\,k'[t]}{1-r^2\,k[t]},\frac{2\,k[t]}{1-r^2\,k[t]}+\frac{a'[t]\left(\left(2-2r^2\,k[t]\right)a'[t]+r^2\,a[t]\,k'[t]\right)}{\left(-1+r^2\,k[t]\right)^2}-\right.$$

$$\left(a[t]\left(-4r^2\left(-1+r^2\,k[t]\right)a'[t]\,k'[t]+4\left(-1+r^2\,k[t]\right)^2a''[t]+\right.\right.$$

$$\left.\left.a[t]\left(3r^4\,k'[t]^2-2r^2\left(-1+r^2\,k[t]\right)k''[t]\right)\right)\right)\Big/\left(4\left(-1+r^2\,k[t]\right)^3\right),0,0\Big\},$$

$$\left\{0,0,\frac{1}{-2+2r^2\,k[t]}r^2\left(4r^2\,k[t]^2-4a'[t]^2-r^2\,a[t]\,a'[t]\,k'[t]-\right.\right.$$

$$\left.\left.2a[t]\,a''[t]+2k[t]\left(-2+2r^2\,a'[t]^2+r^2\,a[t]\,a''[t]\right)\right),0\right\},$$

$$\left\{0,0,0,\frac{1}{-2+2r^2\,k[t]}r^2\,\mathrm{Sin}[\theta]^2\left(4r^2\,k[t]^2-4a'[t]^2-r^2\,a[t]\,a'[t]\,k'[t]-\right.\right.$$

$$\left.\left.2a[t]\,a''[t]+2k[t]\left(-2+2r^2\,a'[t]^2+r^2\,a[t]\,a''[t]\right)\right)\right\}\Big\}$$

In[102]:= **SCurvature[met, x]**

Out[102]=
$$\frac{1}{2\,a[t]^2\left(-1+r^2\,k[t]\right)^2}\Big(12\,r^4\,k[t]^3+12\,a'[t]^2+8\,r^2\,a[t]\,a'[t]\,k'[t]+12\,r^2\,k[t]^2\left(-2+r^2\,a'[t]^2+r^2\,a[t]\,a''[t]\right)+a[t]\left(3\,r^4\,a[t]\,k'[t]^2+12\,a''[t]+2\,r^2\,a[t]\,k''[t]\right)-2\,k[t]\left(-6+12\,r^2\,a'[t]^2+4\,r^4\,a[t]\,a'[t]\,k'[t]+12\,r^2\,a[t]\,a''[t]+r^4\,a[t]^2\,k''[t]\right)\Big)$$

In[103]:= **EinsteinTensor[met, x]**

Out[103]=
$$\Big\{\Big\{\frac{3\,r^2\,k[t]^2+3\,k[t]\left(-1+r^2\,a'[t]^2\right)-a'[t]\left(3\,a'[t]+r^2\,a[t]\,k'[t]\right)}{a[t]^2\left(-1+r^2\,k[t]\right)},\ \frac{r\,k'[t]}{1-r^2\,k[t]},\ 0,\ 0\Big\},$$
$$\Big\{\frac{r\,k'[t]}{1-r^2\,k[t]},\ \frac{k[t]+a'[t]^2+2\,a[t]\,a''[t]}{-1+r^2\,k[t]},\ 0,\ 0\Big\},$$
$$\Big\{0,\ 0,\ -\frac{1}{4\left(-1+r^2\,k[t]\right)^2}r^2\Big(4\,r^4\,k[t]^3+4\,a'[t]^2+6\,r^2\,a[t]\,a'[t]\,k'[t]+4\,r^2\,k[t]^2\left(-2+r^2\,a'[t]^2+2\,r^2\,a[t]\,a''[t]\right)+a[t]\left(3\,r^4\,a[t]\,k'[t]^2+8\,a''[t]+2\,r^2\,a[t]\,k''[t]\right)-2\,k[t]\left(-2+4\,r^2\,a'[t]^2+3\,r^4\,a[t]\,a'[t]\,k'[t]+8\,r^2\,a[t]\,a''[t]+r^4\,a[t]^2\,k''[t]\right)\Big),\ 0\Big\},$$
$$\Big\{0,\ 0,\ 0,\ -\frac{1}{4\left(-1+r^2\,k[t]\right)^2}r^2\,\mathrm{Sin}[\theta]^2\Big(4\,r^4\,k[t]^3+4\,a'[t]^2+6\,r^2\,a[t]\,a'[t]\,k'[t]+4\,r^2\,k[t]^2\left(-2+r^2\,a'[t]^2+2\,r^2\,a[t]\,a''[t]\right)+a[t]\left(3\,r^4\,a[t]\,k'[t]^2+8\,a''[t]+2\,r^2\,a[t]\,k''[t]\right)-2\,k[t]\left(-2+4\,r^2\,a'[t]^2+3\,r^4\,a[t]\,a'[t]\,k'[t]+8\,r^2\,a[t]\,a''[t]+r^4\,a[t]^2\,k''[t]\right)\Big)\Big\}\Big\}$$

In[104]:= **SqRicci[met, x]**

Out[104]=
$$\frac{1}{16\,a[t]^4\left(-1+r^2\,k[t]\right)^4}\Bigg(32\,r^2\,a[t]^2\left(-1+r^2\,k[t]\right)^3k'[t]^2+8\left(-1+r^2\,k[t]\right)^2\Big(-4\,r^2\,k[t]^2+4\,a'[t]^2+r^2\,a[t]\,a'[t]\,k'[t]+2\,a[t]\,a''[t]-2\,k[t]\left(-2+2\,r^2\,a'[t]^2+r^2\,a[t]\,a''[t]\right)\Big)^2+a[t]^2\Big(4\,r^2\left(-1+r^2\,k[t]\right)a'[t]\,k'[t]-12\left(-1+r^2\,k[t]\right)^2a''[t]+a[t]\left(-3\,r^4\,k'[t]^2+2\,r^2\left(-1+r^2\,k[t]\right)k''[t]\right)\Big)^2+16\left(-1+r^2\,k[t]\right)^6\Bigg(\frac{2\,k[t]}{1-r^2\,k[t]}+\frac{a'[t]\left(\left(2-2\,r^2\,k[t]\right)a'[t]+r^2\,a[t]\,k'[t]\right)}{\left(-1+r^2\,k[t]\right)^2}-\frac{1}{4\left(-1+r^2\,k[t]\right)^3}a[t]\Big(-4\,r^2\left(-1+r^2\,k[t]\right)a'[t]\,k'[t]+4\left(-1+r^2\,k[t]\right)^2a''[t]+a[t]\left(3\,r^4\,k'[t]^2-2\,r^2\left(-1+r^2\,k[t]\right)k''[t]\right)\Big)\Bigg)^2\Bigg)$$

In[105]:= **SqRiemann[met, x]**

Out[105]=

$$\frac{1}{16\, a[t]^4}\Bigg(64\left(k[t]+a'[t]^2\right)^2+\frac{64\, r^2\, a[t]^2\, k'[t]^2}{-1+r^2\, k[t]}+\frac{32\left(2\, r^2\, k[t]^2+2\, k[t]\left(-1+r^2\, a'[t]^2\right)-a'[t]\left(2\, a'[t]+r^2\, a[t]\, k'[t]\right)\right)^2}{\left(-1+r^2\, k[t]\right)^2}+128\, a[t]^2\, a''[t]^2+\frac{1}{\left(-1+r^2\, k[t]\right)^4}3\, a[t]^2\left(-4\, r^2\left(-1+r^2\, k[t]\right)a'[t]\, k'[t]+4\left(-1+r^2\, k[t]\right)^2 a''[t]+a[t]\left(3\, r^4\, k'[t]^2-2\, r^2\left(-1+r^2\, k[t]\right)k''[t]\right)\right)^2+\frac{1}{\left(-1+r^2\, k[t]\right)^4}a[t]^2\left(4\, r^2\left(-1+r^2\, k[t]\right)a'[t]\, k'[t]-4\left(-1+r^2\, k[t]\right)^2 a''[t]+a[t]\left(-3\, r^4\, k'[t]^2+2\, r^2\left(-1+r^2\, k[t]\right)k''[t]\right)\right)^2\Bigg)$$

The above "functions" do perform a **Simplify[]** on the result, but you may want to further manipulate the expression into something that may be more
usable, depending on the particular application.
Note: Einstein Equations employed here is *R*ab--gab*R=8GTab , where misses a minus sign, and the Ricci tensor is defined a little different from the one uesed in the Textbook, there is one more minus sign in fron t of the definition. So in the end the result remains the same.