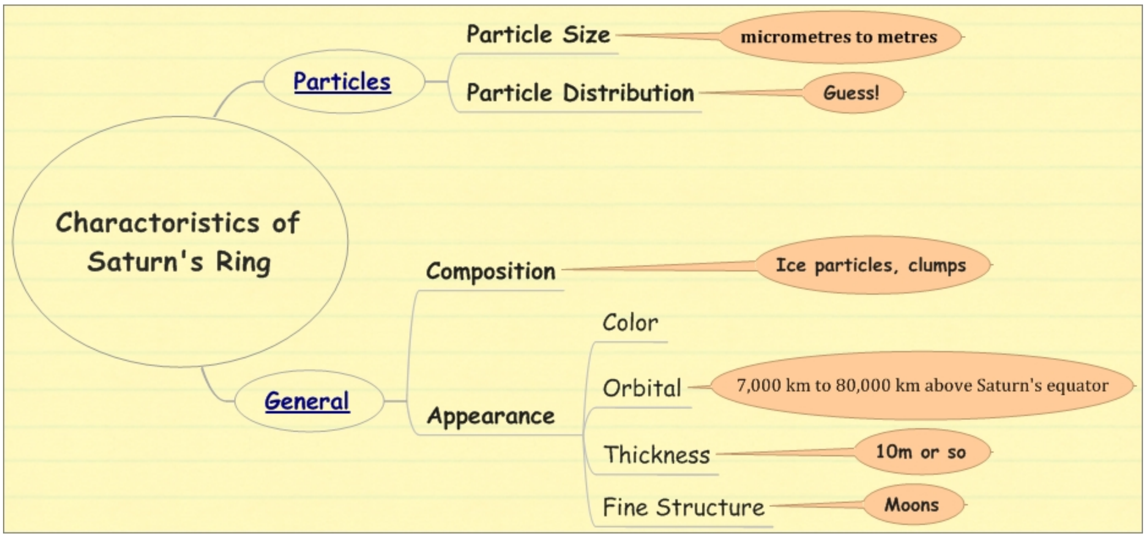
A Theoretical View of Saturn' s Ring

Abstract

The characteristics of Saturn's ring system found up to press time were listed. Theory of the instable band and particle distibution were given.

Characteristics

Galileo is the first human on the earth to observe the rings with a telescope. Then Robert Hooke. Then Cassini found the gaps thus the first one is called the Cassini Division. Nowadays, with more powerfull telescopes and flying by detectors, we have gained more details.
Roughly speaking, what we found out about Saturn' s ring system in the past centuries can be summurized in to several aspects. Most of the features can be summarized into this chart.



Since Maxwell or rather to say Jeffreys has already ruled out gas or liquid structure with very simple arguements, the following work I am going to show is base on a particle ring model. The particle size is from micrometers to meters. However, the distribution of them is unknown. So we have to guass! Generally, the ingrediant is ice with little dirts. The orbital of it is about 7000km to 80000km above saturn's equaor. An amaziong thing is that it is possess a thickness of meters.

The details of the diagram are shown in the following.
A. Micro properties, i.e., details about those particles.
1. Particle size.
Particles vary from micrometres to metres.[1].
2. Particle distribution.
The rings are roughly less denser and possess smaller particles when their orbital radii become smaller. But it is really a complex distribution if we look into its fine structure.
B. Marcro properties.
1. Particle ingredients.
Those particles are mainly composed of ice with little tholins or silicates. [2]
2. Color.
It is generally ashen seen from Cassini spacecraft's natural color photo obtained on May 9, 2007. [3]
3. Optical depth.
This is important because we can get infomation about the general distribution of matter by testing it. The outcome is that the ring is subdivided into ringlets which are of high optical depth and gaps which are of low optical depth.
4. Thickness.
It is 10m or so which means the ring is one particle thick disk. [4][5][6]
5. Orbital.
Typical radius of rings are about 1.2×10^8 meters and typical angular speed in rings is $3 \times 10^{-4}\ s^{-1}$.[7] Most of the rings are almost perfect circles except a few rings. [8][9]
6. Fine structure.
This characteristic is important for disturbance indicates additional condensation objects otherwise we have to find out a intrinsic mechanism for this phenomenon such as a theory the same as the one for spiral arm of galaxies. [10][11][12] It is unexplained that the B ring has a spokes structure.
Finally, what should be pointed out is that the structure infomation of the rings can not be completely detected by only probe photons of visible light. Other optical band as well as other physical effects (i.e., interactions) such as gravitational effect should be involved to depict the rings' physical existence. In this point of view, our knowledge of Saturn's rings is far from complete. Thus the charactristics in the foregoing statement is only part of the truth.

Physical Considerations

To come up with a theoretical explanation, we have to find out the physical meanings of the charateristics.
To interpret the instablity of Saturn's ring, other works have to be down first.
From the physics sense, the particle size, particle distribution (as well as the thickness of the rings) should be examined before we go to the fine structure because the instability here we are going to check is based on those characteristics.
As for the size of the particles, tidal force and the composition of the particles should be condsidered. The competetion between tidal force which related to the gradient of gravational force and the inner tensile force which related to the meterials that makes up the particles decides the maxium radius of the particles. Another consideration is that small particles will snowball into a larger one under the gravation between each other. This reminds us that the size of particles will stay in a narrow band around the maximum radius. However, they are not. [13][16] It is obvious that a orbiting moon can clear out a gap of the particles. One example for it is the Encke Gap inside the A ring cleared out by Pan. [14] Moons can also cause other effects such as existance of a narrow ringlet in a gap.
For the distribution of the particles, Roche limit, beyond which the particles would aggregate into large balls, decides the maxim radius which the grains reach. Statistical methods should be applied to calculate the general particle distribution so as to explain the disk like ring system.
disturbances inside the ringlets such as waves are mostly explained by the perturbance of nearby moons. The spokes structure in B ring remains unexplained as stated in former section.

Theoretical Explanations

The main phenomenon we are going to explain is the general distribution of the particle. As we already know the rings are made of nearly pure ice particles, the discussions such as whether the rings are solid circles or fluid or gaseous or particulate done by Laplace and Maxwell et al. will not be shown here. [15] From now on, the derivations will be down under the knowledge of particulate.
A perfect theory is to set up the complete equations discribing the movements of the particles applying Newton's law and calculate the exact distribution using statistal physics. However, on one hand, we have too little infomation about the composite of the ring; on the other hand, this kind of theory gives little obvious physics. So the following part of this paper only involves several phenomenological theory and tightly bonded to observations. More generally is is a hydrodynamical method.

Particle size

First thing first, the particle size should be derived. The model to be established is based on the fact that the particle size distribution is very board. [16] With several facts listed in Goldreich & Tremaine's paper [15](No.1 and No.7 on page 256 and 257), it is a good approximation the we use the power-law form of the relative surface number density of particles

$$\frac{\mathrm{dN}(R)}{d(\log R)} = \frac{1}{R_0^2}\left(\frac{R}{R_0^2}\right)^{-p}$$

A detailed explanation is done in Peter Goldreich et al.'s work. In their work, the 1st point of the constraints means that the number density of the system has dilation sysmmetry under at least two different scales. This indicates that the system might be scale free under any scales. The 2nd point a normalization constrain. The 3rd point is a defination. It gives a effective radius if the particles are all the same. The area shaded by particles over the optical depth as well as the volume of the particles in a area over the area could lead us to such a concept. A different constant 3/4 multiplied to the defination make no sense. The 7th point adds credibility to our assumption.
Apply those constraints given by Goldreich & Tremaine (1982), an result of the simple model is p=3, $R_0 = 4$cm, $R_{\text{min}} = 13$cm, $R_{\text{max}} = 200$m, $R_\Sigma = 0.9$m.
This is really a good result when comparing with the data I listed in the beginning of this article.

Particle distribution

Because of tidal force, the maximum particle radius at a certain orbital differs. Classically, regardless of the tensorial force inside the orbiting object, Roche limit reads

$$r_L = R\left(\frac{2\rho_M}{\rho_m}\right)^{1/3}, \text{ or } r_L = 1.523\left(\frac{M_p}{\rho}\right)^{1/3} \text{ in our condition,}$$

in which R is the radius of the primary object, ρ_M is the density of the primary object and ρ_m is the density of the orbiting satelite. It involves no orbital radius!
However, the tensile strength T becomes much more important than the self-gravity. Thus the particle radius should satisfy

$$\frac{32}{19}\,\frac{\text{GM}}{r^3}\,\rho \text{R}_s < T$$

where R_s is the radius of the satellite, r is the radius of the orbit, M is the mass of the primary object, ρ is the density of the satellite. Solve the equation we get
$R_s < r^3 \times 10^{-19}\, m^{-2}$.
Since r ranges from 10^7m to 10^8m, we can get the radius limit of the particle at any orbits. Generally, it is less than 10^5m, which is coincide with observations.
The thickness of the rings can be evaluated with the equipartition of energy theorem. Finally we get a answer of the mean square thickness is about 100 times of the particle radius. A good physical explaination is shown in [15].
However there are theories show that the many-particle-thick model appears to be feasible.[17] But here we do not concern much about this because this article is concentrated on the physics in the ring system.

Instability

Finally we come to the stablility problem.
There are several ways to investigate this rather complex system. However, one simple but profound way is to view the particles as a fluid and do the calculation using fluid mechanics. This is valid because the collisions are rather frequent that the particles seemed to possess a random motion as the thermal vibration of fluid molecules and particles are viscous coupled as the the interaction between molecules. Experiments done by P. Evesque and J. Rajchenbach in 1996 [18] show that particle systems can be treated as fluid in some cases. In our ring system the particle size is much smaller than the dimension of the whole system and there are viscous in the system which is similar with fluid. So we will try it.
To be more simple, we use a thin Keplerian particle disk model which means that all small particles in the system is cirling with a Keplerian motion. This is a good approximation because the change of the orbital radius of the particles induced by collisions etc. is very small compared to the average radius of the partilces.
First, to have a precise view of the flow of particles, we define the particle drift flux, $\dot{n} = \frac{-2}{\Omega R} \frac{\partial g}{\partial R}$, in which $g \equiv 3\pi\Omega R^2 \Sigma\nu$ is the viscous couple, Ω is the orbital angular velocity, ν is the kinematic viscosity, Σ is the surface number density of the disk. Negative flux means particles rush inward, vice versa. Physically, if g increase with R, particles move inward to avoid too much dissipation under viscous stress.
We adopt $\sigma \propto \Sigma^\delta$ (power-law), in which σ is the velocity dispersion.
To be more clear, rewrite $\dot{n} = B \frac{\partial\Sigma}{\partial R}$, in which $B \equiv \frac{-2}{(\Omega R)} \partial g / \partial\Sigma$. This formula is valid when the perturbation is axisymmetric with a wavelength W<<R.

$$\partial g/\partial\Sigma>0 \Rightarrow B<0 \Rightarrow \text{stable}$$
$$\partial g/\partial\Sigma<0 \Rightarrow B>0 \Rightarrow \text{instable}$$

This formula gives us nothing but a list of unclear parameters. To have more insight into this, do some approximation on g,

$$\partial g / \partial\Sigma \approx \partial(\Sigma\nu)/\partial\Sigma = {\sigma_0}^{2\delta-1}\, \partial\left(\Sigma^{2\delta}\left(1+\tau^{-2}\right)\tau^{-1}\right) \Big/ \partial\Sigma$$

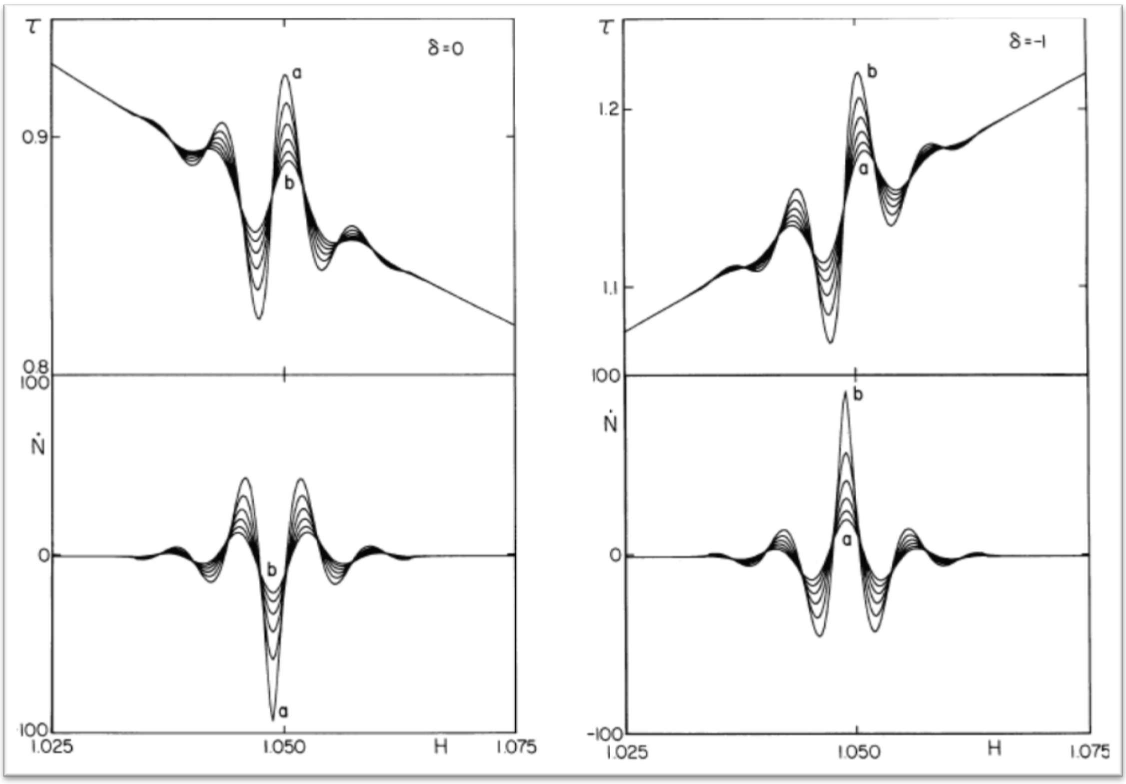
in which ν is substituted by $\nu = \sigma^2 \Big/ \left(\Omega\left(\tau + \tau^{-1}\right)\right)$.
In this way, we connected the power-law index with the stability.

$$\delta<0 \text{ for } \tau>1 \Rightarrow \text{unstable}$$
$$\delta<-1 \text{ for } \tau<<1 \Rightarrow \text{unstable}$$

for sufficiently low Σ (or τ), g must vanish with Σ because other parameters in g can not go to infinity. => $\partial g/\partial\Sigma>0$ => stable
for sufficiently high Σ (or τ), g goes to infinity with Σ because other parameters can not go to zero. => $\partial g/\partial\Sigma>0$ => stable
To be accurate, other parameters is used to derive equations which is useless in our discussion.

Numerical calcultion

To show the theory, Lin et al. [19]did numerical experiments.



Meaning of symbols:
τ=ΣA, Σ is related to δ (power-law exponent)
$\dot{N} = \dot{n} / (3\pi\Sigma\nu)$, $\dot{n} = \frac{-2}{\Omega R}\, \partial g / \partial R$, flux of particle flow (minus means inward)
$H = (R / R_0)^{1/2}$, in which R_0 is difined as $\tau(R_0)$=1. (We can alway normalize the observed τ, i.e., optical depth)
In the figures, a is the initial and b is the end. The six curves separated by equal time intervals and total simulated time is e-folding time. (The can not do a longer simultion than this because the numerical truncation errors grow!)
I will give the explanations of the two figures:
Fig. 1 shows the evolution of a smal perturbation when δ=0 (which means Σ decrease with R). redefined $\dot{N}$ becomes small as time goes on.
Fig. 2 is a unstable condition.
Both of the simulations have no cutoff even they need one when Σ is too large because power-law is assumed for $\sigma \propto \Sigma^{\delta}$. But this does not affect the trend that $\dot{N}$ evolves. Another important argue is that $\dot{N}$ is the flux of particle flow. Even $\dot{N}$ is cut when it is too high, particles still flow to other places as long as $\dot{N}$ is nonzero.
From the simulation we know that the theory is relatively complete because we can reach different phenomena through adjusting the parameters.
The most important thing we know from the theory is that ringlets can form from a slightly perturbated ring through viscous interaction in a Keplerian particles disk.
Figure 2 gives us an impression that the flow of particle may exist forever! That is because this simulation is done under perturbation and Lin did not give cutoff to the system! Physics concerning gives a tendency that when the particles becomes very little, the flow trend to stop.

Outlook

In the real ring system, observations show that at the gaps τ<<1. This indicates that δ<1 in these regions.
However, there are many problems to be looked at. In the theory interpreted is not time dependent, so the time dependent structure - spokes structure for example - is can not be examined.
Another problem is that axisymmetric perturbation is not introduced also. This is another problem that leads the unexplained spokes structure.
The third problem is that the origin of the power law is not explained.
As I wrote in the former part, the instability is not just the particle flow instability. The inelastic collisions are consuming the energy of the system, thus as time goes, the radius of the ring would shrink. The persistency of the ringlets which is another kind of instability is discussed in Lin's paper.[19] In other words, since dissipation always exist, the source of the energy must be explained or the system would die (crash to Saturn) as time go on. That is another work, namely the lifetime of Saturn's ring.

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References

1. http : // www.nasa.gov/worldbook/saturn_worldbook.html
2. Nicholson, P.D. and 16 co-authors. Icarus 193: 182– 212 (2008).
3. http://photojournal.jpl.nasa.gov/catalog/PIA08389
4. Lane, A.L.. et al. Science 215, 537-543 (1982).
5. Zebker, H. A. & Tyler, G. L. Science (1984).
6. http:/ / www. sciencedaily. com/releases/ 2005/ 11/ 051110220809. htm
7. Null (1976)
8. Porco, C.; Nicholson, P. D.; Borderies, N.; Danielson, G. E.; Goldreich, P.; Holberg, J. B.; Lane, A. L. Icarus 60 (1): 1– 16 (1984).
9. Porco, C. C.; Nicholson, P. D.. Icarus 72 (2): 437– 467 (1987).
10. Hedman, Matthew M.; Burns, Joseph A,; Showalter, Mark R. at al.. Icarus 188: 89– 107 (2007).
11. Weiss, J. W., Porco, C. C., Tiscareno, M. S.. The Astronomical Journal 138 (1): 272– 286 (2009).
12. Porco, C.C., Baker, E., Barbara, J., et al.. Science 307 (5713): 1226– 1236 (2005).
13. Cuzzi & Pollack (1978).
14. Esposito, L. W.. Reports on Progress in Physics 65: 1741– 1783(2002).
15. Peter Goldreich, Scott Tremaine. Ann. Rev. Astron. Astrophys. 20: 249-83(1982).
16. Cuzzi et al. (1980)
17. Jeffery N., Cuzzi, Joseph A. Burns, Richard H. Durisen & Patrck M. Hamill. Nature 281, 202 - 204 (1979).
18. P. Evesque and J. Rajchenbach, Phys. Rev. Lett 12, 623 (1996).
19. D. N. C. Lin, P. Bodenheimer. The Astrophysical Journal, 248: L83-L86 (1981).