Modifying Gravity - f(R) Gravity
Outline

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1 References & Conventions

Reference: **arXiv**:astro-ph/0611321; **arXiv**:1002.3868; **arXiv**:astro-ph/9910176

Conventions

- $' = \frac{\partial}{\partial \tau}$
- $f \equiv f(R)$
- $f_R \equiv \frac{\partial f}{\partial R}$
- $\mathcal{H} = a'/a$
- $\mathrm{d}s^2 = a(\tau)^2(\mathrm{d}\tau^2 + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j)$ is the line element used throughout this note.
- $R = \frac{6a''}{a^3}$ is the scalar curvature.

2 Main Equations

2.1 Jordan frame

Start from action

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}[R + f(R)] + \int \mathrm{d}^4\sqrt{-g}\mathcal{L}_{(M)}(x_i, g_{\mu\nu}) \tag{1}$$

Variation with respect to $g_{\mu\nu}$ of this action gives [1]

$$(1 + f_R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R + f) + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_R = \kappa^2 T_{\mu\nu} \tag{2}$$

Or

$$(1 + f_R)G_{\mu\nu} + \frac{1}{2}g_{\mu\nu}(f_R R - f) + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_R = \kappa^2 T_{\mu\nu} \tag{3}$$

[1] The standard procedure is really complicate. Ohanian et al has a somewhat easier method in his book. The used a special set of coordinates. $g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}$, then things become easier. Palatini method should be done with another action.

Background Background equations are

$$\left|\begin{array}{l}(1+r_R)\mathcal{H}^2+\frac{a^2}{6}f-\frac{a''}{a}f_R+\mathcal{H}f_R'=\frac{\kappa^2}{3}a^2\rho\\ \frac{a''}{a}-(1+f_R)\mathcal{H}^2+a^2\frac{f}{6}+\mathcal{H}f_R'+\frac{1}{2}f_R''=-\frac{\kappa^2}{6}a^2(\rho+3P)\end{array}\right. \tag{4}$$

Unknown variables: f/f_R, $\mathcal{H}$, ρ. f should be given in a certain model.

Perturbations Metric

$$\left|\begin{array}{l}g_{00}=-a^2(1+2AY)\\ g_{0i}=-a^2BY_i\\ g_{ij}=a^2(\gamma_{ij}+2H_LY\gamma_{ij}+2H_TY_{ij})\end{array}\right. \tag{5}$$

H_T is the anisotropic distortion of each constant time hypersurface. H_L is the trace part.
E-M tensor

$$\left|\begin{array}{l}T^0{}_0=-\rho(1+\delta Y)\\ T^0{}_i=(\rho+p)(v-B)Y_i\\ T^i{}_0=-(\rho+p)vY^i\\ T^i{}_j=[p\delta^i_j+\delta p\delta^i_jY+\frac{3}{2}(\rho+p)\sigma Y^i_j]=p[\gamma^i{}_j+\pi_L\delta^i{}_j+\pi_TY^i_j]\end{array}\right. \tag{6}$$

v is the potential of velocity, $v^i\equiv u^i/u^0=vY^i$. Subscripts T means tranverse, or simply traceless part. Subscripts L means Longitudinal, or trace part.

Conservation equations (calculated from ${T_{(\lambda)}}^{\nu}{}_{\mu;\nu}$[2]. However we can define some kind of density and pressure and include these in the conservatioon equation. AND this won't affect my work since I don't really use this equation here. The conservation equation are derived in the way I stated here.)

$$\left|\begin{array}{l}\delta'+(1+w)(kv+3H_L')+3\mathcal{H}(\frac{\delta p}{\delta\rho}-w)\delta=0\\ (v'-B)+\mathcal{H}(1-3w)(v-B)+\frac{w'}{1+w}(v-B)-\frac{\delta p/\delta\rho}{1+w}k^2\delta-k^2A+\frac{2}{3}k^2\sigma=0\end{array}\right. \tag{7}$$

Conservation equations themselves are not enough.

We have the defination of $\delta R/Y$

$$\left|\ \frac{\delta R}{Y}=\frac{2}{a^2}\left[-6\frac{a''}{a}A-3\mathcal{H}A'+k^2A+kB'+3k\mathcal{H}B+9\mathcal{H}H_L'+3H_L''+2k^2(H_L+\frac{H_T}{3})\right]\right. \tag{8}$$

Using the standard procedure given by Kodama et al, we can find the perturbation equations. In *Cosmologia Notebook - 2012-02*, Page 18.

Also we can transform them into what they are in Synchronous Gauge. In *Cosmologia Notebook - 2012-02*, Page 18, 19.

2.2 From Jordan Frame to Einstein Frame

Why this transformation **arXiv**:astro-ph/9910176 mentioned "Jordan frame formulation of a scalar-tensor theory is not viable because the energy density of the gravitational scalar field present in the theory is not bounded from below", thus violating the weak energy condition[3]. So I would

[2] Why the same with SGR? Actually I thought the conservation law should be something from the identity that ${G_{ab}}^{;a}=0$. This DOESN'T lead to the conclustion that ${T_{ab}}^{;a}=0$

[3] Weak energy condition: for timelike vector field U^α, $\rho=T_{\alpha\beta}U^\alpha U^\beta >= 0$

like to work in Einstein frame though Einstein frame also has problems such as a violation of equivalence priciple[4].

Action in Jordan frame is given by

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^4x\sqrt{-g}[R + f(R)] + \int \mathrm{d}^4\sqrt{-g}\mathcal{L}_{(M)}(x_i, g_{\mu\nu}) \tag{9}$$

Apply a gauge transformation

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{10}$$

we get the action in Einstein frame.

$$\begin{aligned} \tilde{S} = \frac{1}{2\kappa^2} \int \mathrm{d}^4x\sqrt{-\tilde{g}}\tilde{R} + \int \mathrm{d}^4x\sqrt{-\tilde{g}}[-\tfrac{1}{2}\tilde{g}^{\mu\nu}(\tilde{\nabla}_\mu\phi)(\tilde{\nabla}_\nu\phi) - V(\phi)] \\ + \int \mathrm{d}^4x\sqrt{-\tilde{g}}e^{-2\beta\kappa\phi}\mathcal{L}_{(M)}(x_i, e^{-\beta\kappa\phi}\tilde{g}_{\mu\nu}) \end{aligned} \tag{11}$$

[Definations of $V(\phi)$, β, ϕ, $e^{-2\omega}$, $\Omega^2 \equiv e^{2\omega(x^\alpha)}$, in *Cosmologia Notebook - 2012-02*, Page 23.]

Here I write down the simplified potential[5] $V(\phi) = \frac{Rf_R - f}{2\kappa^2(1+f_R)^2}$. Given a explicit model, this will be determined and may posses order 2 of ϕ.

Then variation gives field equation

$$\tilde{G}_{\mu\nu} = \kappa^2\tilde{T}_{\mu\nu} + \tfrac{1}{2}\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi + \tfrac{1}{2}(\tilde{g}^{\alpha\gamma}\tilde{\nabla}_\alpha\phi\tilde{\nabla}_\gamma\phi)\tilde{g}_{\mu\nu} - V(\phi)\tilde{g}_{\mu\nu} \tag{12}$$

$$\begin{aligned} &\tilde{T}_{\mu\nu} = (\tilde{\rho} + \tilde{P})\tilde{U}_\mu\tilde{U}_\nu + \tilde{p}\tilde{g}_{\mu\nu} \\ &\tilde{U}_\mu \equiv e^{\beta\kappa\phi/2}U_\mu \\ &\tilde{\rho} = e^{-2\beta\kappa\phi}\rho \\ &\tilde{p} \equiv e^{-2\beta\kappa\phi}p \end{aligned} \tag{13}$$

Trace of field equation,

$$G = \kappa^2 T + \tfrac{1}{2}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi + 2\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi - 4V(\phi) \tag{14}$$

$$V_\phi \equiv \frac{\mathrm{d}V}{\mathrm{d}\phi} \tag{15}$$

2.3 Einstein Frame

Background Background equations are

(From conservation equation and Field equation? I didn't derive them myself.)

$$\begin{aligned} &\phi'' + 2\tilde{\mathcal{H}}\phi' + \tilde{a}^2V_\phi = \tfrac{1}{2}\kappa\beta\tilde{a}^2(\tilde{\rho} - 3\tilde{p}) \\ &\tilde{\rho}' + 3\tilde{\mathcal{H}}(\tilde{\rho} + \tilde{p}) = -\tfrac{1}{2}\kappa\beta\phi'(\tilde{\rho} - 3\tilde{p}) \end{aligned} \tag{16}$$

Field equations are

$$\begin{aligned} &\tilde{\mathcal{H}}^2 = \tfrac{1}{3}\kappa^2(\tfrac{1}{2}\phi'^2 + \tilde{a}^2V(\phi) + \tilde{a}^2\tilde{\rho}_c + \tilde{a}^2\rho_\gamma) \\ &\phi'' + 2\tilde{\mathcal{H}}\phi' + \tilde{a}^2V_\phi = \tfrac{1}{2}\kappa\beta\tilde{a}^2\tilde{\rho}_c \\ &\tilde{\rho}_c \equiv \tilde{\rho}_c^* e^{-\kappa\beta\phi/2} \\ &\tilde{\rho}_c^* = \tilde{\rho}_c^{*0}/\tilde{a}^3 \end{aligned} \tag{17}$$

[4]equality of inertial mass and gravitational mass is equivalent to the assertion that the acceleration imparted to a body by a gravitational field is independent of the nature of the body, $m_I \cdot a = Grav \cdot m_G$.

[5]The decompostion of $\phi(\tau) + \delta\phi(\vec{x}, \tau)$ is used and assume the background ϕ only evolves with time τ.

Since for radiation $\tilde{p} = \frac{\tilde{\rho}}{3}$[6], from conservation equations

$$\tilde{\mathcal{H}}' - \tilde{\mathcal{H}}^2 = -\frac{1}{2}\kappa^2(\phi'^2 + \tilde{\rho}_c + \frac{4}{3}\tilde{\rho}_\gamma) \tag{18}$$

$$\tilde{\rho}_\gamma' + 4\tilde{\mathcal{H}}\tilde{\rho}_\gamma = 0 \tag{19}$$

The actual equation for matter is

$$\tilde{\rho}_c' + 3\tilde{\mathcal{H}}\tilde{\rho}_c = -Const \cdot \beta\kappa\phi'\tilde{\rho}_c \tag{20}$$

Unkown variables: ϕ, $\tilde{\mathcal{H}}$, $\tilde{\rho}_c$, $\tilde{\rho}_\gamma$, $\tilde{p}$. These two equations are just some of the complete equation system. We have to use Field equations to form a complete system.

[Defination of δ_c and θ_c, $\delta_c = \frac{\delta\tilde{\rho}_c^*}{\tilde{\rho}_c^*}$. In *Cosmologia Notebook - 2012-02*, Page 24.] The scalar field is decomposed into $\phi(t) + \delta\phi(\vec{x}, t)$.

Perturbations Perturbation equations are

$$\begin{aligned} &\tilde{\delta}_c'' + \tilde{\mathcal{H}}\tilde{\delta}_c' - \frac{3}{2}\tilde{\mathcal{H}}^2(2\tilde{\Omega}_\gamma\tilde{\delta}_\gamma + \tilde{\Omega}_c(\tilde{\delta}_c - \frac{1}{2}\kappa\beta\delta\phi) + 2\kappa^2\phi'\delta\phi' - \kappa^2 V_\phi\delta\phi) = 0 \\ &\delta\phi'' + 2\tilde{\mathcal{H}}\delta\phi' + k^2\delta\phi + \tilde{a}^2 V_{,\phi\phi}\delta\phi - \phi'\tilde{\delta}_c' - \frac{3}{2}\frac{\beta}{\kappa}\tilde{\mathcal{H}}^2\tilde{\Omega}_c(\tilde{\delta}_c - \frac{1}{2}\kappa\beta\delta\phi) = 0 \\ &\tilde{\delta}_\gamma'' + \frac{1}{3}k^2\tilde{\delta}_\gamma - \frac{1}{3}\tilde{\delta}_c'' = 0 \end{aligned} \tag{21}$$

And then it is possible to solve these equations since there are only 3 variables unkown, $\tilde{\delta}_c$, $\tilde{\delta}_\gamma$, $\tilde{\delta}\phi$.

Then we can discuss the equations in matter dominated era. In *Cosmologia Notebook - 2012-02*, Page 26. (It is about the attractor point.)

$$\begin{aligned} &\tilde{\delta}_c'' + \frac{3}{2}\frac{1}{\tau}\tilde{\delta}_c' - 3\frac{1}{\tau^2}(\tilde{\delta}_c - \frac{1}{2}\kappa\beta\delta\phi) + \frac{2\kappa}{\beta}\frac{1}{\tau}\delta\phi' = 0 \\ &\delta\phi'' + 2\tilde{\mathcal{H}}\delta\phi' + k^2\delta\phi - \frac{1}{\beta\kappa}\frac{1}{\tau}\tilde{\delta}_c' - \frac{3\beta}{\kappa}\frac{1}{\tau^2}(\tilde{\delta}_c - \frac{1}{2}\kappa\beta\delta\phi) = 0 \end{aligned} \tag{22}$$

[6] This is interesting because we have such transformations: $\tilde{\rho} \equiv e^{-2\beta\kappa\phi}\rho$ and $\tilde{p} \equiv e^{-2\beta\kappa\phi}p$. (Also the velocity transformation is $\tilde{U}_\mu \equiv e^{\beta\kappa\phi/2}U_\mu$ thus we can define a consitent E-M tensor which has the same form as in Jordan frame in Einstein frame.)